AXIOMS FOR HOMOLOGY

Eilenberg, Steenrod, and Milnor obtained a system of axioms, which characterize homology theories without bothering with simplices & singular chains.

DEFINITION

A **HOMOLOGY THEORY** is a map that assigns the following:

(I) $\forall$ pair of spaces (X, A) a graded abelian group $H_p(X, A)$, $p \in \mathbb{Z}$.

(II) $\forall$ $(X, A), (Y, B)$ and $f: (X, A) \to (Y, B)$, a homomorphism $f_*: H_p(X, A) \to H_p(Y, B)$ $\forall p \in \mathbb{Z}$.

(III) $\forall$ pair of spaces (X, A), a homomorphism $\partial_*: H_p(X, A) \to H_{p-1}(A)$, $\forall p \in \mathbb{Z}$.

Notation: $H_p(X, \emptyset)$ is denoted by $H_p(X)$.

Furthermore:

⊛ the assigments in (I) & (II) are assumed to be functorial, ie.

$\forall (X,A), (Y,B), (Z,C)$ and for all maps

$$(X,A) \xrightarrow{f} (Y,B), \ (Y,B) \xrightarrow{g} (Z,C)$$

we have $(g \circ f)_* = g_* \circ f_*$, and

for $id: (X,A) \to (X,A)$ we have

$$(id_*) = id_{H_p(X,A)} \ \forall p.$$

⊛ The homomorphisms

$\partial_* : H_p(X,A) \to H_{p-1}(A)$ are assumed to be natural, ie $\forall$ map $f: (X,A) \to (Y,B)$ the following diagram commutes

$$\begin{tikzcd}
H_p(X,A) \arrow[r, "\partial_*"] \arrow[d, "f_*"'] \arrow[dr, phantom, "©"] & H_{p-1}(A) \arrow[d, "(f|_A)_*"] \\
H_p(Y,B) \arrow[r, "\partial_*"] & H_{p-1}(B)
\end{tikzcd}$$

All the above is assumed to satisfy the following axioms:

① HOMOTOPY INVARIANCE

If $f \simeq g$ as maps $(X,A) \to (Y,B)$ then $f_* = g_* : H_p(X,A) \to H_p(Y,B)$.

② EXACTNESS AXIOM

$\forall$ pair (X,A) consider $i : A \to X$ (the inclusion) and $j : (X,\phi) \to (X,A)$ (coming from $id : X \to X$), then the sequence

$$\cdots \xrightarrow{\partial_*} H_p(A) \xrightarrow{i_*} H_p(X) \xrightarrow{j_*} H_p(X,A) \xrightarrow{\partial_*} H_{p-1}(A) \xrightarrow{i_*} \cdots$$

is exact.

③ EXCISION THEOREM

Let (X,A) be a pair, $U \subset X$ an open subset with $\overline{U} \subset \mathring{A}$. Let

$k: (X \setminus U, A \setminus U) \hookrightarrow (X, A)$

be the inclusion. then

$k_*: H_p(X \setminus U, A \setminus U) \to H_p(X, A)$ is

an isomorphism for all p.

④ DIMENSION AXIOM

$H_i(\text{point}) = 0 \quad \forall i \neq 0.$

⑤ ADDITIVITY

disjoint union

Let $X = \bigsqcup_{\alpha \in I} X_\alpha$, and $i_\alpha : X_\alpha \to X$

the inclusions $\Rightarrow$

$$\bigoplus_{\alpha \in I} (i_\alpha)_* : \bigoplus H_p(X_\alpha) \xrightarrow{\cong} H_p(X)$$

is an isomorphism $\forall p$.

DEFINITION

Let H be homology theory. then

$G := H_0(\text{point})$ is called the coefficient group of the theory.

(in our examples, we had $G = \mathbb{Z}$).